

HYPERSURFACES WITH DEFECT

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ABSTRACT. A projective hypersurface $X \subseteq \mathbb{P}^n$ has *defect* if $h^i(X) \neq h^i(\mathbb{P}^n)$ for some $i \in \{n, \dots, 2n-2\}$ in a suitable cohomology theory. This occurs for example when $X \subseteq \mathbb{P}^4$ is not \mathbb{Q} -factorial. We show that in characteristic 0, the Tjurina number of hypersurfaces with defect is large. For X with mild singularities, there is a similar result in positive characteristic. As an application, we obtain a lower bound on the asymptotic density of hypersurfaces without defect over a finite field.

1. INTRODUCTION

Let K be a field and let $n \geq 3$ be an integer. A projective hypersurface $X \subseteq \mathbb{P}_K^n$ is said to have *defect* if

$$h^i(X) \neq h^i(\mathbb{P}_K^n) \quad \text{for some } i \in \{n, \dots, 2n-2\},$$

where h^n denotes the n -th Betti number in a reasonable cohomology theory for K -varieties. Some prominent examples of such cohomology theories include:

- singular, algebraic de Rham or Kähler-de Rham cohomology if K is of characteristic zero,
- rigid cohomology if K is a perfect field of positive characteristic,
- étale cohomology.

In any of these theories, a hypersurface with defect is necessarily singular. Moreover, it seems that defect forces the hypersurface to have “many” singularities compared to their degree: For example, an important class of hypersurfaces with defect is formed by non-factorial hypersurfaces $X \subseteq \mathbb{P}^4$ (see also Section 4). By a result of Cheltsov [7], if such an X has at most ordinary double points as singularities, then the singular locus consists of at least $(\deg(X) - 1)^2$ nodes.

Another family of hypersurfaces of defect in \mathbb{P}^n is given by cones over smooth hypersurfaces in \mathbb{P}^{n-1} , see Corollary 2.16 and the subsequent remark. The vertex of the cone is a singularity with big Milnor number.

The literature on defect (e.g., [11], [33], [36]) is mainly on hypersurfaces with at most ordinary double points as singularities and exclusively over the field of complex numbers. The aim of this paper is to generalize the philosophy “defect implies many singularities” to arbitrary projective hypersurfaces over arbitrary fields. In Section 2, we prove the following:

Theorem 1.1. *Let K be a field of characteristic zero. Suppose that $X \subseteq \mathbb{P}_K^n$, $n \geq 3$, is a hypersurface with defect in algebraic de Rham, Kähler-de Rham, singular or étale cohomology.*

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Denote by $\tau(X)$ the global Tjurina number of X . Then

$$\tau(X) \geq \frac{\deg(X) - n + 1}{n^2 + n + 1}.$$

Moreover, if X has at most weighted homogeneous singularities, then

$$\tau(X) \geq \deg(X) - n + 1.$$

Of course, $\tau(X)$ will only be finite if X has at most isolated singularities. The main ingredient in the proof is a close inspection of the algebraic de Rham cohomology of hypersurface complements in the spirit of Griffiths [16] and Dimca [11].

The situation for positive characteristic fields is much more subtle. As explained in Subsection 2.11, there are some obstructions to extending the proof of Theorem 1.1. However, for hypersurfaces with very mild singularities, we can use a resolution of singularities approach similar to [31] to prove:

Theorem 1.2. *Let K be an algebraically closed field of characteristic $\neq 2$. Let $X \subseteq \mathbb{P}_K^n$ be a hypersurface with defect in étale or rigid cohomology. Suppose further that X has a zero-dimensional singular locus $\Sigma = \Sigma_O \cup \Sigma_A$, where*

- Σ_O is formed by $x \in \Sigma$ being ordinary multiple points of multiplicity m_x and
- Σ_A consists of $x \in \Sigma$ which are singular points of type A_{k_x} .

Then

$$\sum_{x \in \Sigma_O} m_x + \sum_{x \in \Sigma_A} 2 \left\lceil \frac{k_x}{2} \right\rceil \geq \deg(X).$$

For details, see Section 3. We conjecture that the theme “defect implies many singularities” should extend to arbitrary hypersurfaces in any positive characteristic.

As an application of Theorem 1.2, we prove in Section 5 (Corollary 5.5) a lower bound on the density of hypersurfaces without defect over a finite field:

Theorem 1.3. *Let q be an odd prime power. Then*

$$\lim_{d \rightarrow \infty} \frac{\#\{f \in \mathbb{F}_q[x_0, \dots, x_n]_d \mid \{f = 0\} \subseteq \mathbb{P}_{\mathbb{F}_q}^n \text{ has no defect}\}}{\#\mathbb{F}_q[x_0, \dots, x_n]_d} \geq \frac{1}{\zeta_{\mathbb{P}^n}(n+3)} = \prod_{i=3}^{n+2} (1 - q^{-i}).$$

In view of Theorem 1.1 and [26, Corollary 5.9], we believe that this limit is actually 1.

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2. DE RHAM COHOMOLOGY

2.1. Two de Rham cohomology theories. Let K be a field of characteristic zero and let X be a K -variety. Suppose that X admits a closed embedding $X \hookrightarrow Y$ into some smooth K -variety Y . One can associate two related cohomology theories to X coming from Kähler differentials:

- *Kähler-de Rham cohomology* ([1]): This is the hypercohomology of the de Rham complex on X :

$$H_{\text{KdR}}^\bullet(X) := \mathbb{H}^\bullet(X, \Omega_X^\bullet).$$

- *Algebraic de Rham cohomology* ([18]): This is defined as the hypercohomology of the formal completion of the de Rham complex on Y along X :

$$H_{\text{dR}}^\bullet(X \hookrightarrow Y) := \mathbb{H}^\bullet(Y/X, \Omega_{Y/X}^\bullet).$$

If \mathcal{I}_X denotes the ideal sheaf of X in Y , the natural projection

$$\Omega_{Y/X}^\bullet = \varprojlim_j \Omega_Y^\bullet / \mathcal{I}_X^j \rightarrow \Omega_Y^\bullet / \mathcal{I}_X \rightarrow \Omega_X^\bullet$$

is compatible with the exterior derivative and is hence a morphism of complexes. Taking hypercohomology, this gives rise to a natural functorial comparison map

$$H_{\text{dR}}^\bullet(X \hookrightarrow Y) \rightarrow H_{\text{KdR}}^\bullet(X).$$

By [18, Theorem II.1.4], the algebraic de Rham cohomology of X does not depend on the embedding $X \hookrightarrow Y$, and we will hence simply write $H_{\text{dR}}^\bullet(X)$. In particular, if X is smooth, the above comparison map is an isomorphism.

In general, algebraic de Rham cohomology always gives the “correct” Betti numbers in the following sense:

Theorem 2.1 ([18, Theorem IV.1.1]). *Suppose $K = \mathbb{C}$. Then there is a natural isomorphism*

$$H_{\text{dR}}^\bullet(X) \cong H_{\text{sing}}^\bullet(X^{\text{an}})$$

between the algebraic de Rham cohomology of X and the singular cohomology of the associated analytic space X^{an} .

However, for singular X , Kähler-de Rham cohomology tends to give bigger Betti numbers:

Theorem 2.2 ([6, Corollary 3.14]). *Suppose $K = \mathbb{C}$ and that X is complete or has at most isolated singularities. Then $H_{\text{sing}}^\bullet(X^{\text{an}}, \mathbb{C})$ is a direct summand of $H_{\text{KdR}}^\bullet(X)$.*

Note that by the Lefschetz principle, we can always find an embedding of K into \mathbb{C} , and both cohomology theories are compatible with field extensions.

2.2. Cohomological tools. For future reference, we briefly mention some standard facts and tools. The notation $H^i(X)$ without any subscript refers to either algebraic de Rham or Kähler-de Rham cohomology. The dimension of $H^i(X)$ as a K -vector space will be denoted by $h^i(X)$.

Fact 2.3 (Betti numbers of affine and projective space). *Let $n \geq 1$ be an integer. Then*

$$h^i(\mathbb{A}_K^n) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0 \end{cases} \quad \text{and} \quad h^i(\mathbb{P}_K^n) = \begin{cases} 1 & \text{if } i \in \{0, 2, \dots, 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Fact 2.4 (Excision). *Let Y be a K -variety and let $X \subseteq Y$ be a closed subscheme. Then there is a long exact sequence*

$$\cdots \rightarrow H^i(Y) \rightarrow H^i(Y \setminus X) \rightarrow H_X^{i+1}(Y) \rightarrow H^{i+1}(Y) \rightarrow \cdots,$$

where $H_X^\bullet(Y)$ denotes local cohomology of Y with support in X . Moreover, if $U \subseteq Y$ is an open subscheme containing X , then there is a natural isomorphism $H_X^i(Y) \cong H_X^i(U)$.

Fact 2.5 (Smooth Gysin sequence). *Suppose that K is algebraically closed. Let Y be a smooth K -variety and let $X \subseteq Y$ be a closed smooth subvariety of codimension r . Then there is a long exact sequence*

$$\cdots \rightarrow H^i(Y) \rightarrow H^i(Y \setminus X) \xrightarrow{\rho} H^{i+1-2r}(X) \rightarrow H^{i+1}(Y) \rightarrow \cdots$$

The map ρ is called Poincaré residue map.

Proof. This is the content of [17, Theorem III.8.3]. \square

For algebraic de Rham cohomology, there is a good theory $H_{c,dR}^i$ of cohomology with compact support such that $H_{c,dR}^i(X) = H_{dR}^i(X)$ whenever X is proper [3].

Fact 2.6 (Compact support Gysin sequence). *Let Y be an arbitrary K -variety and let $X \subseteq Y$ be a closed subscheme. Then there is a long exact sequence*

$$\cdots \rightarrow H^i(Y)_{c,dR} \rightarrow H_{c,dR}^i(X) \rightarrow H_{c,dR}^{i+1}(Y \setminus X) \rightarrow H_{c,dR}^{i+1}(Y) \rightarrow \cdots$$

Fact 2.7 (Poincaré duality). *Let Y be a smooth K -variety of dimension n and let X be a closed subscheme. Then there is a perfect pairing*

$$H_{c,dR}^\bullet(X) \times H_{dR,X}^{2n-\bullet}(Y) \rightarrow K.$$

In particular $h^i(Y) = h_{c,dR}^{2n-i}(Y)$ and if Y is proper, then $h^i(Y) = h^{2n-i}(Y)$ for all i .

In the case of algebraic de Rham cohomology of smooth varieties, the smooth Gysin sequence arises as Poincaré dual of the compact support Gysin sequence.

Fact 2.8 (Cohomological dimension). *For any K -variety X , $H^i(X) = 0$ and $H_{c,dR}^i(X) = 0$ for $i < 0$ and $i > 2 \dim X$.*

Fact 2.9 (Cohomological dimension of affines). *Let X be affine of dimension n . Then $H_{dR}^i(X) = 0$ for $i > n$. Moreover, if $X \subseteq \mathbb{A}_K^{n+1}$ is a hypersurface, then $H_{KdR}^i(X) = 0$ for $i > n$.*

Proof. The general result on algebraic de Rham cohomology follows from Theorem 2.1 and the corresponding vanishing for Stein spaces [6, Corollary 3.15].

Suppose now that X is a hypersurface in affine $(n+1)$ -space. Denote by R the coordinate ring of \mathbb{A}_K^{n+1} and suppose that the hypersurface X is defined by $f \in R$. Consider the natural surjection

$$\Omega_R^\bullet \rightarrow \Omega_R^\bullet \otimes R/(f) \rightarrow \Omega_{R/(f)}^\bullet.$$

This is compatible with the exterior derivative d and gives thus a short exact sequence

$$0 \rightarrow \mathcal{K}^\bullet \rightarrow \Omega_R^\bullet \rightarrow \Omega_{R/(f)}^\bullet \rightarrow 0$$

of complexes. This yields in turn a long exact sequence in cohomology, which reads

$$\cdots \rightarrow H^i(\mathcal{K}^\bullet) \rightarrow H^i(\mathbb{A}^{n+1}) \rightarrow H_{KdR}^i(X) \rightarrow H^{i+1}(\mathcal{K}^\bullet) \rightarrow \cdots$$

Since $\Omega_R^i = 0$ for $i > n+1$, we have $\mathcal{K}^i = 0$ and $\Omega_{R/(f)}^i = 0$ and thus $H_{KdR}^i(X) = 0$ for $i > n+1$. Moreover, $H^n(\mathbb{A}^{n+1}) = 0$ and $\mathcal{K}^{n+2} = 0$ imply $H_{KdR}^{n+1}(X) = 0$. \square

2.3. Algebraic de Rham cohomology of projective hypersurfaces. Consider the projective space \mathbb{P}_K^n over K . Let $\overline{X} \subseteq \mathbb{P}_K^n$ be a hypersurface. Then the first half of the algebraic de Rham cohomology of \overline{X} is well understood:

Lemma 2.10 (Lefschetz hyperplane theorem). *The natural restriction $H_{\text{dR}}^i(\mathbb{P}^n) \rightarrow H_{\text{dR}}^i(\overline{X})$ is an isomorphism for $i \leq n - 2$ and injective for $i \leq n - 1$.*

Proof. The restriction map fits into the Gysin sequence with compact support:

$$\cdots \rightarrow H_{c,\text{dR}}^i(\mathbb{P}^n \setminus \overline{X}) \rightarrow H_{\text{dR}}^i(\mathbb{P}^n) \rightarrow H_{\text{dR}}^i(\overline{X}) \rightarrow H_{c,\text{dR}}^{i+1}(\mathbb{P}^n \setminus \overline{X}) \rightarrow \cdots$$

By Poincaré duality,

$$H_{c,\text{dR}}^i(\mathbb{P}^n \setminus \overline{X}) \cong H^{2n-i}(\mathbb{P}^n \setminus \overline{X})^\vee.$$

The variety $\mathbb{P}^n \setminus \overline{X}$ is smooth and affine of dimension n . Hence by Fact 2.9 we conclude that $H^{2n-i}(\mathbb{P}^n \setminus \overline{X})$ vanishes for $2n - i > n$, i.e. $i < n$. \square

If \overline{X} is smooth, then the Lefschetz hyperplane theorem combined with Poincaré duality on \overline{X} gives almost all the Betti numbers:

Corollary 2.11. *Suppose that \overline{X} is smooth. Then $h^i(\overline{X}) = h^i(\mathbb{P}^n)$ for $i \notin \{n - 1, 2n\}$.*

Due to dimension reasons, $h^{2n}(\overline{X}) = 0$. The middle Betti number of a smooth hypersurfaces can be computed by the methods of Griffiths [16].

However, for singular hypersurfaces, Poincaré duality may fail. From now on, we will focus on the case of isolated singularities, i.e. the singular locus of \overline{X} has dimension 0.

Lemma 2.12. *Suppose that \overline{X} has only isolated singularities. Then*

- $h_{\text{dR}}^i(\overline{X}) = h^i(\mathbb{P}^n)$ for $i \notin \{n - 1, n, 2n\}$,
- $h_{\text{KdR}}^i(\overline{X}) = h^i(\mathbb{P}^n)$ for $n + 1 \leq i \leq 2n - 1$.

Proof. Again by dimension reasons, $h_{\text{dR}}^{2n}(\overline{X}) = 0$, whereas $h^{2n}(\mathbb{P}^n) = 1$. Denote by Σ the singular locus of \overline{X} . By Bertini's theorem, after possibly extending the base field, there is a hyperplane $H \subseteq \mathbb{P}^n$ such that $\Sigma \cap H = \emptyset$ and $\overline{Y} := \overline{X} \cap H$ is a smooth hypersurface in $H \cong \mathbb{P}^{n-1}$. In particular

$$h^i(\overline{Y}) = h^i(\mathbb{P}^{n-1}) = h^i(\mathbb{P}^n), \quad i \leq n - 3.$$

Let $X := \overline{X} \setminus \overline{Y}$. This is a singular hypersurface in \mathbb{A}^n , so $H^i(X) = 0$ for $i \geq n$. Using the long exact sequence

$$\cdots \rightarrow H^{i-1}(X) \rightarrow H_{\overline{Y}}^i(\overline{X}) \rightarrow H^i(\overline{X}) \rightarrow H^i(X) \rightarrow \cdots,$$

we obtain

$$H_{\overline{Y}}^i(\overline{X}) \cong H^i(\overline{X}), \quad i \geq n + 1.$$

Since \overline{Y} is a smooth closed subscheme of $\overline{X} \setminus \Sigma$, excision and Poincaré duality on $\overline{X} \setminus \Sigma$ yield

$$h_{\overline{Y},\text{dR}}^i(\overline{X}) = h_{\overline{Y},\text{dR}}^i(\overline{X} \setminus \Sigma) = h^{2n-2-i}(\overline{Y}).$$

Thus

$$h_{\text{dR}}^i(\overline{X}) = h^{2n-2-i}(\mathbb{P}^n) = h^i(\mathbb{P}^n), \quad i \geq n + 1.$$

For Kähler-de Rham cohomology, we have again $H_{\text{KdR}}^i(\overline{X}) \cong H_{Y, \text{KdR}}^i(\overline{X} \setminus \Sigma)$ for $i \geq n+1$. Note that $H_{\overline{Y}}^i(\overline{X} \setminus \Sigma)$ fits into a long exact sequence

$$\dots \rightarrow H^{i-1}(\overline{X} \setminus \Sigma) \rightarrow H^{i-1}(X \setminus \Sigma) \rightarrow H_{\overline{Y}}^i(\overline{X} \setminus \Sigma) \rightarrow H^i(\overline{X} \setminus \Sigma) \rightarrow H^i(X \setminus \Sigma) \rightarrow \dots$$

in both cohomology theories. In particular, since $\overline{X} \setminus \Sigma$ and $X \setminus \Sigma$ are smooth,

$$\begin{aligned} h_{\text{KdR}}^i(\overline{X}) &= h_{Y, \text{KdR}}^i(\overline{X} \setminus \Sigma) \\ &= \dim \ker (H^i(\overline{X} \setminus \Sigma) \rightarrow H^i(X \setminus \Sigma)) + \dim \text{coker} (H^{i-1}(\overline{X} \setminus \Sigma) \rightarrow H^{i-1}(X \setminus \Sigma)) \\ &= h_{Y, \text{dR}}^i(\overline{X} \setminus \Sigma) \\ &= h_{\text{dR}}^i(\overline{X}), \quad i \geq n+1. \end{aligned}$$

The case $i \leq n-2$ in algebraic de Rham cohomology is handled by the Lefschetz hyperplane theorem 2.10. \square

2.4. Defect. Let $\overline{X} \subseteq \mathbb{P}_K^n$ be a hypersurface with at most isolated singularities.

- The *defect of \overline{X} in algebraic de Rham cohomology* is $\delta_{\text{dR}}(\overline{X}) := h_{\text{dR}}^n(\overline{X}) - h_{\text{dR}}^n(\mathbb{P}^n)$.
- The *defect of \overline{X} in Kähler-de Rham cohomology* is $\delta_{\text{KdR}}(\overline{X}) := h_{\text{KdR}}^n(\overline{X}) - h_{\text{KdR}}^n(\mathbb{P}^n)$.

Clearly $\delta_{\text{dR}}(\overline{X}) \leq \delta_{\text{KdR}}(\overline{X})$ by Theorem 2.2. We will show in Corollary 2.15 that in fact equality holds. Hence we can simply speak of *defect* and denote it by $\delta(\overline{X})$. Furthermore, we will say that \overline{X} *has defect* if $\delta(\overline{X}) > 0$.

Remarks. More remarks on defect:

- In other words, \overline{X} has defect if the n -th Betti numbers of \overline{X} and \mathbb{P}^n do not agree. In particular, a hypersurface \overline{X} with defect has to be singular.
- Since \overline{X} is assumed to have at most isolated singularities, $h^i(\overline{X}) = h^i(\mathbb{P}^n)$ for all $i \neq \{n-1, n, 2n\}$ by Lemma 2.12.
- By Lemma 2.13, the defect of \overline{X} is always non-negative.
- Defect depends only on the Betti numbers. Since our cohomology theories involved are compatible with field extensions, we may as well assume that K is algebraically closed.

2.5. Defect and cokernels. In the remainder of this subsection, we will give some cohomological characterizations of defect for hypersurfaces with isolated singularities. Let $n \geq 3$ be an integer and let $\overline{X} \subseteq \mathbb{P}_K^n$ be a hypersurface with singular locus Σ . Assume that $\dim \Sigma = 0$. Again by Bertini's theorem, we can find a hyperplane $H \subseteq \mathbb{P}_K^n$ such that $H \cap \Sigma = \emptyset$ and $\overline{X} \cap H$ is smooth. Define $X := \overline{X} \setminus (\overline{X} \cap H)$; this is a singular hypersurface in $\mathbb{P}^n \setminus H \cong \mathbb{A}^n$.

Lemma 2.13. *Consider the long exact sequence*

$$\dots \rightarrow H^{n-1}(\overline{X} \setminus \Sigma) \xrightarrow{\alpha} H_{\Sigma}^n(\overline{X}) \rightarrow H^n(\overline{X}) \rightarrow H^n(\overline{X} \setminus \Sigma) \rightarrow H_{\Sigma}^{n+1}(\overline{X}) \rightarrow \dots$$

Then $\delta(\overline{X}) = \dim \text{coker } \alpha$.

Proof. The proof consists of a few technical computations. We first assume that $n \geq 4$.

- $\overline{X} \setminus \Sigma$ is a smooth closed subvariety of codimension one in $\mathbb{P}^n \setminus \Sigma$. The corresponding Gysin sequence is

$$\cdots \rightarrow H^{n+1}(\mathbb{P}^n \setminus \overline{X}) \rightarrow H^n(\overline{X} \setminus \Sigma) \rightarrow H^{n+2}(\mathbb{P}^n \setminus \Sigma) \rightarrow H^{n+2}(\mathbb{P}^n \setminus \overline{X}) \rightarrow \cdots$$

Since $\mathbb{P}^n \setminus \overline{X}$ is smooth and affine of dimension n , there is an isomorphism

$$H^n(\overline{X} \setminus \Sigma) \cong H^{n+2}(\mathbb{P}^n \setminus \Sigma).$$

Σ is a closed subvariety of codimension n in \mathbb{P}^n . The associated Gysin sequence is

$$\cdots \rightarrow H_{\text{dR}}^{n-2}(\Sigma) \rightarrow H^{n-2}(\mathbb{P}^n) \rightarrow H_{c,\text{dR}}^{n-2}(\mathbb{P}^n \setminus \Sigma) \rightarrow H_{\text{dR}}^{n-3}(\Sigma) \rightarrow \cdots$$

Since $n \geq 4$, we can use $\dim \Sigma = 0$ to obtain

$$h^{n+2}(\mathbb{P}^n \setminus \Sigma) = h_{c,\text{dR}}^{n-2}(\mathbb{P}^n \setminus \Sigma) = h^{n-2}(\mathbb{P}^n).$$

This shows that $h^n(\overline{X} \setminus \Sigma) = h^{n-2}(\mathbb{P}^n) = h^n(\mathbb{P}^n)$.

- Since Σ lies inside the affine part $X \subseteq \overline{X}$, $H_{\Sigma}^{n+1}(X) = H_{\Sigma}^{n+1}(\overline{X})$. Using the two Gysin sequences for $X \setminus \Sigma \subseteq \mathbb{A}^n \setminus \Sigma$ and $\Sigma \subseteq \mathbb{A}^n$ gives

$$h^n(X \setminus \Sigma) = h^{n+2}(\mathbb{A}^n \setminus \Sigma) = h^{n-2}(\mathbb{A}^n) = 0.$$

On the other hand, $H^{n+1}(X) = 0$ since X is an affine hypersurface of dimension $n - 1$. The excision sequence for $\Sigma \subseteq X$ then yields $H_{\Sigma}^{n+1}(X) = 0$.

- Putting this together,

$$h^n(\overline{X}) = \dim \operatorname{coker} \alpha + h^n(\overline{X} \setminus \Sigma) = \dim \operatorname{coker} \alpha + h^n(\mathbb{P}^n).$$

- In the case $n = 3$, the long exact sequence in the statement of the lemma gives

$$h^3(\overline{X}) = \dim \operatorname{coker} \alpha + h^3(\overline{X} \setminus \Sigma) - h_{\Sigma}^4(\overline{X}) + h^4(\overline{X}) - h^4(\overline{X} \setminus \Sigma),$$

where we used $H_{\Sigma}^5(\overline{X}) = 0$ for dimension reasons. Using Poincaré duality on $\overline{X} \setminus \Sigma$ and the compact support Gysin sequence

$$0 \rightarrow H_{c,\text{dR}}^0(\overline{X} \setminus \Sigma) \rightarrow H_{\text{dR}}^0(\overline{X}) \rightarrow H_{\text{dR}}^0(\Sigma) \rightarrow H_{c,\text{dR}}^1(\overline{X} \setminus \Sigma) \rightarrow H_{\text{dR}}^1(X) = 0$$

we obtain

$$h^3(\overline{X} \setminus \Sigma) - h^4(\overline{X} \setminus \Sigma) = h_{c,\text{dR}}^1(\overline{X} \setminus \Sigma) - h_{c,\text{dR}}^0(\overline{X} \setminus \Sigma) = h_{\text{dR}}^0(\Sigma) - h_{\text{dR}}^0(\overline{X}).$$

Note that $h_{\text{dR}}^0(\overline{X}) = 1$ and $h^4(\overline{X}) = 1$ by Lemma 2.12. Thus

$$h^3(\overline{X}) = \dim \operatorname{coker} \alpha + h_{\text{dR}}^0(\Sigma) - h_{\Sigma}^4(\overline{X}).$$

Moreover,

$$h_{\Sigma}^4(\overline{X}) = h_{\Sigma}^4(X) = h^3(X \setminus \Sigma) = h_{c,\text{dR}}^1(X \setminus \Sigma) = h_{\text{dR}}^0(\Sigma),$$

the last step uses the compact support Gysin sequences

$$\cdots \rightarrow H_{c,\text{dR}}^0(X) \rightarrow H_{\text{dR}}^0(\Sigma) \rightarrow H_{c,\text{dR}}^1(X \setminus \Sigma) \rightarrow H_{c,\text{dR}}^1(X) \rightarrow \cdots$$

and

$$0 = H_{c,\text{dR}}^i(\mathbb{A}^n) \rightarrow H_{c,\text{dR}}^i(X) \rightarrow H_{c,\text{dR}}^{i+1}(\mathbb{A}^3 \setminus X) = 0, \quad i = 0, 1.$$

Consequently

$$\delta(\overline{X}) = h^3(\overline{X}) = \dim \operatorname{coker} \alpha. \quad \square$$

The open immersion $X \hookrightarrow \overline{X}$ induces a commutative ladder

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^{n-1}(\overline{X}) & \longrightarrow & H^{n-1}(\overline{X} \setminus \Sigma) & \xrightarrow{\alpha} & H_{\Sigma}^n(\overline{X}) \longrightarrow H^n(\overline{X}) \longrightarrow \cdots \\
& & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & H^{n-1}(X) & \xrightarrow{\vartheta} & H^{n-1}(X \setminus \Sigma) & \longrightarrow & H_{\Sigma}^n(X) \longrightarrow H^n(X) \longrightarrow \cdots
\end{array}$$

of long exact sequences.

Lemma 2.14. *We have $\delta(\overline{X}) = \dim \operatorname{coker} \beta$, where*

$$\beta : H^{n-1}(\overline{X} \setminus \Sigma) \rightarrow H^{n-1}(X \setminus \Sigma) / \vartheta(H^{n-1}(X))$$

is the map induced by $X \setminus \Sigma \hookrightarrow \overline{X} \setminus \Sigma$.

Proof. As $H^n(X) = 0$, the natural map $H^{n-1}(X \setminus \Sigma) \rightarrow H_{\Sigma}^n(X)$ is surjective. Since its kernel is given by the image of ϑ ,

$$H^{n-1}(X \setminus \Sigma) / \vartheta(H^{n-1}(X)) \rightarrow H_{\Sigma}^n(X)$$

is an isomorphism. The singular locus Σ lies inside the affine part X , so the natural map $H_{\Sigma}^n(\overline{X}) \rightarrow H_{\Sigma}^n(X)$ is an isomorphism as well. Therefore $\operatorname{coker} \beta \cong \operatorname{coker} \alpha$, which finishes the proof by the preceding Lemma 2.13. \square

Corollary 2.15. *Let $\overline{X} \subseteq \mathbb{P}^n$ be a hypersurface with at most isolated singularities. Then $\delta_{\mathrm{dR}}(\overline{X}) = \delta_{\mathrm{KdR}}(\overline{X})$. In particular $h_{\mathrm{dR}}^n(\overline{X}) = h_{\mathrm{KdR}}^n(\overline{X})$.*

Proof. By Theorem 2.2, it remains to show the inequality $\delta_{\mathrm{dR}}(\overline{X}) \geq \delta_{\mathrm{KdR}}(\overline{X})$. To this end, note that the comparison map between algebraic and Kähler-de Rham cohomology yields a commutative diagram

$$\begin{array}{ccc}
H_{\mathrm{dR}}^{n-1}(X) & \xrightarrow{\vartheta_{\mathrm{dR}}} & H_{\mathrm{dR}}^{n-1}(X \setminus \Sigma) \\
\downarrow & & \downarrow \simeq \\
H_{\mathrm{KdR}}^{n-1}(X) & \xrightarrow{\vartheta_{\mathrm{KdR}}} & H_{\mathrm{KdR}}^{n-1}(X \setminus \Sigma).
\end{array}$$

This gives a surjection

$$H_{\mathrm{dR}}^{n-1}(X \setminus \Sigma) / \vartheta_{\mathrm{dR}}(H_{\mathrm{dR}}^{n-1}(X)) \twoheadrightarrow H_{\mathrm{KdR}}^{n-1}(X \setminus \Sigma) / \vartheta_{\mathrm{KdR}}(H_{\mathrm{KdR}}^{n-1}(X)).$$

If $\beta_{\mathrm{dR}}, \beta_{\mathrm{KdR}}$ denote the two versions of the map β of Lemma 2.14, then this gives rise to a surjection $\operatorname{coker} \beta_{\mathrm{dR}} \twoheadrightarrow \operatorname{coker} \beta_{\mathrm{KdR}}$. Hence

$$\delta_{\mathrm{dR}}(\overline{X}) = \dim \operatorname{coker} \beta_{\mathrm{dR}} \geq \dim \operatorname{coker} \beta_{\mathrm{KdR}} = \delta_{\mathrm{KdR}}(\overline{X}).$$

The reverse inequality follows from Theorem 2.2. \square

Corollary 2.16. *Suppose that $H^{n-1}(X) = 0$. Then $h^n(\overline{X}) = h^{n-2}(\overline{X} \setminus X)$.*

Proof. By Lemma 2.14, the number $\delta(X)$ equals the dimension of the cokernel of the restriction map

$$H^{n-1}(\overline{X} \setminus \Sigma) \rightarrow H^{n-1}(X \setminus \Sigma),$$

which fits into a long exact sequence

$$\dots \rightarrow H^{n-1}(\overline{X} \setminus \Sigma) \rightarrow H^{n-1}(X \setminus \Sigma) \rightarrow H_{\overline{X} \setminus X}^n(\overline{X} \setminus \Sigma) \rightarrow H^n(\overline{X} \setminus \Sigma) \rightarrow \dots$$

By Poincaré duality, $h_{\overline{X} \setminus X}^n(\overline{X} \setminus \Sigma) = h^{n-2}(\overline{X} \setminus X)$. By the proof Lemma 2.13, $h^n(\overline{X} \setminus \Sigma) = h^n(\mathbb{P}^n)$ and $h^n(X \setminus \Sigma) = 0$. Therefore

$$h^n(\overline{X}) = \delta(\overline{X}) + h^n(\mathbb{P}^n) = h_{\overline{X} \setminus X}^n(\overline{X} \setminus \Sigma) - h^n(\overline{X} \setminus \Sigma) + h^n(\mathbb{P}^n) = h^{n-2}(\overline{X} \setminus X). \quad \square$$

Remark. This gives several examples of hypersurfaces with defect: In particular, if \overline{X} is the cone over a smooth projective hypersurface $Y \subseteq \mathbb{P}^{n-1}$, then $\delta(\overline{X}) = h^{n-2}(Y) - h^n(\mathbb{P}^n)$. For example, any cone over a nonsingular plane curve of positive genus has defect.

We finish this section with another cohomological characterization of defect: Using the smooth Gysin sequences for $\overline{X} \setminus \Sigma \subseteq \mathbb{P}^n \setminus \Sigma$ and $X \setminus \Sigma \subseteq \mathbb{A}^n \setminus \Sigma$ respectively, we get a commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^n(\mathbb{P}^n \setminus \overline{X}) & \longrightarrow & H^{n-1}(\overline{X} \setminus \Sigma) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H^n(\mathbb{A}^n \setminus X) & \xrightarrow{\rho} & H^{n-1}(X \setminus \Sigma) & \longrightarrow & \dots, \end{array}$$

where ρ is the Poincaré residue.

Lemma 2.17. *We have $\delta(\overline{X}) \leq \dim \operatorname{coker} \gamma$, where*

$$\gamma : H^n(\mathbb{P}^n \setminus \overline{X}) \rightarrow H^n(\mathbb{A}^n \setminus X) / \rho^{-1}(\vartheta(H^{n-1}(X)))$$

is the map induced by the open immersion $\mathbb{A}^n \setminus X \hookrightarrow \mathbb{P}^n \setminus \overline{X}$. Moreover, equality holds if n is even.

Proof. One checks that $H^n(\mathbb{A}^n \setminus \Sigma) = H^{n+1}(\mathbb{A}^n \setminus \Sigma) = 0$, so ρ is an isomorphism. We obtain a commutative diagram

$$\begin{array}{ccc} H^n(\mathbb{P}^n \setminus \overline{X}) & \xrightarrow{\sigma} & H^{n-1}(\overline{X} \setminus \Sigma) \\ \gamma \downarrow & & \downarrow \beta \\ H^n(\mathbb{A}^n \setminus X) / \rho^{-1}(\vartheta(H^{n-1}(X))) & \xrightarrow[\rho]{\simeq} & H^{n-1}(X \setminus \Sigma) / \vartheta(H^{n-1}(X)), \end{array}$$

where β is as in Lemma 2.14. Thus

$$\dim \operatorname{coker} \gamma = \dim \operatorname{coker}(\beta \circ \sigma) \geq \dim \operatorname{coker} \beta = \delta(\overline{X}).$$

If n is even, then the map σ is surjective, since the preceding term $H^{n+1}(\mathbb{P}^n \setminus \Sigma)$ in the Gysin sequence vanishes. Hence in this case, $\operatorname{coker}(\beta \circ \sigma) = \operatorname{coker} \beta$. \square

2.6. Differential forms on hypersurface complements. We keep the notations from the previous subsection. Suppose that the hypersurface \overline{X} is defined by the homogeneous polynomial $F \in K[x_0, \dots, x_n]_d$. Moreover, assume that the hyperplane H is given by the vanishing of x_0 . Let $f = F(1, x_1, \dots, x_n) \in K[x_1, \dots, x_n]$ denote the defining polynomial of X in $\mathbb{A}^n = \mathbb{P}^n \setminus \{x_0 = 0\}$.

In view of Lemma 2.17, the defect of \overline{X} may be described by investigating the top-dimensional cohomology of the hypersurface complements $\mathbb{P}^n \setminus \overline{X}$ and $\mathbb{A}^n \setminus X$. Fortunately, these spaces can be explicitly described. Both varieties in question are smooth and affine of dimension n , so their n -th algebraic de Rham cohomology is just a quotient of the module of n -forms on their coordinate rings. More precisely:

Lemma 2.18.

(1) $H^n(\mathbb{P}^n \setminus \overline{X})$ is generated by

$$\left\{ \frac{G\Omega}{F^k} \middle| G \in K[x_0, \dots, x_n]_{kd-n-1}, k \geq 0 \right\},$$

where

$$\Omega := \sum_{i=0}^n (-1)^i x_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n.$$

(2) $H^n(\mathbb{A}^n \setminus X)$ is generated by

$$\left\{ \frac{g\omega}{f^k} \middle| g \in K[x_1, \dots, x_n], k \geq 0 \right\},$$

where $\omega := dx_1 \wedge \dots \wedge dx_n$.

(3) The natural restriction map is given by

$$H^n(\mathbb{P}^n \setminus \overline{X}) \rightarrow H^n(\mathbb{A}^n \setminus X), \quad \left[\frac{G\Omega}{F^k} \right] \rightarrow \left[\frac{g\omega}{f^k} \right],$$

where g is the dehomogenization of G .

Proof. (2) and (3) are immediate. For (1), see e.g. [13, Chapter 6]. □

2.7. Pole-order filtration. In the notation of Lemma 2.17, let $V := \rho^{-1}(\vartheta(H^{n-1}(X)))$. For $k \geq 0$, define the pole-order filtration P^k on $H^n(\mathbb{P}^n \setminus \overline{X})$ resp. $H^n(\mathbb{A}^n \setminus X)/V$ as the image of differential forms of the type $G\Omega/F^k$ resp. $g\omega/f^k$. Since $G\Omega/F^k = FG\Omega/F^{k+1}$ and similarly in the affine case, these are ascending filtrations. Note that these are slightly different to the ones given in Dimca's article [11].

The pole-order filtration gives rise to the k -th graded objects

$$\begin{aligned} \mathrm{Gr}_P^k H^n(\mathbb{P}^n \setminus \overline{X}) &:= P^k H^n(\mathbb{P}^n \setminus \overline{X}) / P^{k-1} H^n(\mathbb{P}^n \setminus \overline{X}), \\ \mathrm{Gr}_P^k H^n(\mathbb{A}^n \setminus X) &:= P^k H^n(\mathbb{A}^n \setminus X) / P^{k-1} H^n(\mathbb{A}^n \setminus X), \quad k \geq 0, \end{aligned}$$

with the convention that $P^{-1} = \{0\}$. The natural restriction

$$\gamma : H^n(\mathbb{P}^n \setminus \overline{X}) \rightarrow H^n(\mathbb{A}^n \setminus X)/V$$

induces maps $\mathrm{Gr}_P^k(\gamma)$ on the corresponding graded objects. In view of Lemma 2.17, there is an immediate corollary:

Corollary 2.19. *If \overline{X} has defect, then there is an integer $k \geq 0$ such that $\mathrm{Gr}_P^k(\gamma)$ is not surjective.*

Set $S := K[x_0, \dots, x_n]$ and $R := K[x_1, \dots, x_n]$. The explicit description of the cohomology groups given in Lemma 2.18 yields a commutative diagram

$$\begin{array}{ccc} S_{kd-n-1} & \longrightarrow & R \\ \downarrow & & \downarrow \varphi \\ \mathrm{Gr}_P^k H^n(\mathbb{P}^n \setminus X) & \xrightarrow{\mathrm{Gr}_P^k(\gamma)} & \mathrm{Gr}_P^k(H^n(\mathbb{A}^n \setminus X)/V) \end{array}$$

for any $k \geq 0$ with surjective vertical arrows and the horizontal arrows being the natural restriction maps. We can actually make the top right corner smaller:

Lemma 2.20. *Let $\varphi : R \rightarrow \mathrm{Gr}_P^k(H^n(\mathbb{A}^n \setminus X)/V)$ be as in the above diagram. Let $J(f)$ denote the ideal in R spanned by the partial derivatives of f . Then:*

- (1) *For $k \geq 2$, the map φ factors through $R/((f) + J(f))$.*
- (2) *For $k = 1$, the map φ factors through $R/((f) + J(f)^3)$.*
- (3) $\mathrm{Gr}_P^0 H^n(\mathbb{P}^n \setminus \overline{X}) = \mathrm{Gr}_P^0(H^n(\mathbb{A}^n \setminus X)/V) = 0$.

Proof. If $g \in (f) + J(f)$, then there are polynomials h_0, \dots, h_n such that

$$g = h_0 f + \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}.$$

The class of $h_0 f \omega / f^k$ vanishes in the graded object Gr_P^k by definition. One computes that

$$d \left((-1)^i \frac{h_i}{f^{k-1}} dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n \right) = (k-1) \cdot h_i \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\omega}{f^k} - \frac{\partial h_i}{\partial x_i} \cdot \frac{\omega}{f^{k-1}}.$$

Hence if $k \geq 2$, we can rewrite the cohomology class of $h_i \frac{\partial f}{\partial x_i} \omega / f^k$ as the class of a differential form with lower pole order. But such classes vanish in the graded object Gr_P^k by definition of the pole-order filtration.

For $k = 0$ observe at first that $\mathrm{Gr}_P^0 H^n(\mathbb{P}^n \setminus \overline{X})$ is generated by $S_{n-1} = 0$. If $h \in R$ is any polynomial, then the above relation for $k = 1$ shows that all forms of the type $\frac{\partial h}{\partial x_i} \omega$ vanish in $\mathrm{Gr}_P^0(H^n(\mathbb{A}^n \setminus \overline{X})/V)$. But any form can be written in this way.

If $k = 1$, we cannot apply the pole-order reduction trick anymore. However, we can use the space V : Let $\eta \in \Omega_R^{n-1}$ be a global $(n-1)$ -form. Then the class of η in $\Omega_{R/(f)}^{n-1}$ lies in the kernel of $d : \Omega_{R/(f)}^{n-1} \rightarrow \Omega_{R/(f)}^n$ if and only if $d\eta = f\xi + \zeta \wedge df$ for some $\xi \in \Omega_R^n, \zeta \in \Omega_R^{n-1}$. Such an η defines a cohomology class in $H^{n-1}(X)$.

In the notation of Lemma 2.14, restricting to the open $X \setminus \Sigma$ via ϑ and applying the inverse of the Poincaré residue map ρ (see [17, Theorem III.8.3]), we get a map

$$\rho^{-1} \circ \vartheta : W := \{\eta \in \Omega_R^{n-1} \mid \exists \xi \in \Omega_R^n, \zeta \in \Omega_R^{n-1} : d\eta = f\xi + \zeta \wedge df\} \rightarrow V, \quad \eta \mapsto \left[\frac{\eta \wedge df}{f} \right].$$

In particular, all forms inside the image of this map will vanish in $\mathrm{Gr}_P^1(H^n(\mathbb{A}^n \setminus X)/V)$.

We will now give a description in terms of polynomials: Write

$$\eta := \sum_{i=1}^n (-1)^i h_i \cdot dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n, \quad h_i \in R.$$

Then

$$\eta \wedge df = \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i} \omega \quad \text{and} \quad d\eta = \sum_{i=1}^n \frac{\partial h_i}{\partial x_i}.$$

Now let $g \in (f) + J(f)^3$. Then

$$g = fh' + \sum_{i,j,k=1}^n h_{ijk} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_k} = fh' + \sum_{i=1}^n \left(\sum_{j,k=1}^n h_{ijk} \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_k} \right) \frac{\partial f}{\partial x_i}, \quad h', h_{ijk} \in R$$

and

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sum_{j,k=1}^n h_{ijk} \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_k} \right) \in J(f).$$

Thus if $(g - fh')\omega = \eta \wedge df$ as above, then $d\eta = \zeta \wedge df$ for some ζ and hence $\eta \in W$. In particular, inside $H^n(\mathbb{A}^n \setminus X)$,

$$\left[\frac{g\omega}{f} \right] = \left[\frac{(g - fh')\omega}{f} \right] + \left[\frac{fh'\omega}{f} \right] = \left[\frac{\eta \wedge df}{f} \right] + [h'\omega] = [\rho^{-1}(\vartheta(\eta))] + 0 \in V.$$

Consequently, the map φ factors through $R/((f) + J(f)^3)$. □

2.8. Defect and Tjurina number. We are now in shape to prove the main theorem of this section.

Theorem 2.21. *Let $\tau := \dim_K K[x_1, \dots, x_n]/((f) + J(f))$ be the global Tjurina number of \overline{X} . If \overline{X} has defect, then*

$$\tau \geq \frac{d - n + 1}{n^2 + n + 1}.$$

Moreover, if the map $\text{Gr}_P^k(\gamma)$ is not surjective for some $k \geq 2$, then

$$\tau \geq kd - n + 1.$$

Proof. Since \overline{X} has defect, there is an integer $k \geq 0$ such that $\text{Gr}_P^k(\gamma)$ is not surjective by Corollary 2.19. Using Lemma 2.20 (3), we can assume that $k \geq 1$.

Assume first that $k \geq 2$. Then the non-surjectivity of some $\text{Gr}_P^k(\gamma)$ implies the non-surjectivity of the natural restriction map $S_{kd-n-1} \rightarrow T(f)$, where $T(f) := R/((f) + J(f))$ denotes the global Tjurina algebra of f . Since $\dim \Sigma = 0$, $T(f)$ is a finite-dimensional K -algebra. Applying Poonen's trick [32, Lemma 2.1(b)] shows that the image of S_i in $T(f)$ strictly increases with i until it fills the whole space. In particular, the restriction map has to be surjective for $i \geq \tau - 1$. From this, one infers that $kd - n - 1 \leq \tau - 2$, whence $\tau \geq kd - n + 1$.

If $k = 1$, then the same argument shows that $\dim_K R/((f) + J(f)^3) \geq d - n + 1$. Using the exact sequences of $T(f)$ -modules

$$0 \rightarrow J(f)^i/J(f)^{i+1} \rightarrow R/((f) + J(f)^{i+1}) \rightarrow R/((f) + J(f)^i) \rightarrow 0$$

for $i = 1, 2$, we obtain

$$\dim_K R/((f) + J(f)^3) = \dim_K J(f)^2/J(f)^3 + \dim_K J(f)/J(f)^2 + \dim_K T(f).$$

Since $J(f)^i/J(f)^{i+1}$ can be generated by n^i elements, it has length at most n^i as $T(f)$ -module. Thus

$$d - n + 1 \leq \dim_K R/((f) + J(f)^3) \leq n^2\tau + n\tau + \tau = (n^2 + n + 1) \cdot \tau. \quad \square$$

2.9. Local computations. In the case that the singularities of \overline{X} are weighted homogeneous, we can use the methods of Dimca [12] to improve the bound of Theorem 2.21:

Lemma 2.22. *In the notations of Lemma 2.20, suppose that \overline{X} has only weighted homogeneous singularities. Then the natural map*

$$\varphi : R \rightarrow \mathrm{Gr}_P^1(H^n(\mathbb{A}^n \setminus X)/V)$$

factors through $R/((f) + J(f))$.

Proof. By the Lefschetz principle, assume that $K \subseteq \mathbb{C}$ and use analytic de Rham cohomology. As in [12, Section 3], the map $\mathrm{Gr}_P^1(\gamma)$ can be described as the natural restriction

$$\mathrm{Gr}_P^1(\gamma) : \mathrm{Gr}_P^1 H^n(\mathbb{P}^n \setminus X) \rightarrow \bigoplus_{x \in \Sigma} \mathrm{Gr}_{P_x}^1 H^n(\Omega_{f,x}^\bullet),$$

where $\Omega_{f,x}^\bullet$ denotes the localization of the holomorphic de Rham complex $\Omega_{\mathbb{C}^n,x}^\bullet$ with respect to f , and P_x is the corresponding local pole-order filtration. In particular, for any $x \in \Sigma$ there is a natural surjection

$$\varphi_x : \mathcal{O}_{\mathbb{C}^n,x} \rightarrow \mathrm{Gr}_{P_x}^1 H^n(\Omega_{f,x}^\bullet), \quad g \mapsto \left[\frac{g}{f} dx_1 \wedge \cdots \wedge dx_n \right].$$

Suppose now that the singularity of \overline{X} at x is contact-equivalent to a weighted homogeneous singularity. Then there is a biholomorphic coordinate change ψ sending x to $(0, \dots, 0)$ such that $f' = \psi(f)$ is a weighted homogeneous polynomial. Moreover, ψ induces an isomorphism of the local Tjurina algebras of f at x and f' at 0, respectively.

Take a polynomial $h \in (f) + J(f)$. Under the natural map

$$R \rightarrow H^n(\Omega_{f,x}^\bullet) \xrightarrow{\cong} H^n(\Omega_{f',0}^\bullet)$$

induced by ψ , the class $[h/f dx_1 \wedge \cdots \wedge dx_n]$ is sent to some $[h'/f' dx'_1 \wedge \cdots \wedge dx'_n]$ with h' lying in the analytic ideal $(f') + J(f') \subseteq \mathcal{O}_{\mathbb{C}^n,0}$. However, the calculation [12, Example 3.6] shows that $[h'/f' dx'_1 \wedge \cdots \wedge dx'_n] = 0$. \square

Applying the same methods as in the proof of Theorem 2.21, this yields:

Corollary 2.23. *Suppose that \overline{X} has at most weighted homogeneous singularities. If \overline{X} has defect, then $\tau \geq d - n + 1$.*

A well-known application is the following:

Corollary 2.24 ([11, Proposition 3.4]). *Suppose that \overline{X} has at most ordinary double points as singularities. If \overline{X} has defect, then $\tau \geq \frac{dn}{2} - n + 1$.*

Proof. The polynomial $f' = x_1^2 + \dots + x_n^2$ is weighted homogeneous of degree 2 with respect to the weights $(1, \dots, 1)$. Since the local cohomology piece $\mathrm{Gr}_{P_0}^k H^n(\Omega_{f',0}^\bullet)$ is spanned by homogeneous forms of degree $2k - n$, it vanishes for $k \neq \frac{n}{2}$. In particular, \overline{X} has no defect if n is odd, as the map $\mathrm{Gr}_P^k(\gamma)$ is always surjective. For even n , defect implies that $\mathrm{Gr}_P^k(\gamma)$ is surjective for $k \neq \frac{n}{2}$ and not surjective for $k = \frac{n}{2}$. This concludes the proof by the second part of Theorem 2.21. \square

Remarks. Let $\overline{X} \subseteq \mathbb{P}^n$ be a nodal hypersurface.

- One can actually show that if \overline{X} has defect and $\dim \overline{X} = 3$, then $\tau \geq (d-1)^2$, see [7] or [24, Theorem 4.1]. The latter proof carries over to higher dimensions.
- For even n , it is conjectured in [24] that $\tau \geq (d-1)^{n/2}$.

2.10. Proof of Theorem 1.1.

Proof. The statements for algebraic de Rham and Kähler-de Rham cohomology follow from Theorem 2.21 and Corollary 2.23. Embedding K into \mathbb{C} using the Lefschetz principle, defect in algebraic de Rham is equivalent to defect in singular cohomology by Theorem 2.1. The étale version is a consequence of Artin's comparison theorem between étale and singular cohomology [2, Theorem 5.2]. \square

2.11. Positive characteristic. If K is a field of characteristic $p > 0$, both algebraic and Kähler-de Rham cohomology behave pathologically. For example, affine space has infinite Betti numbers. To remedy this, one needs a different kind of cohomology theory. Of course ℓ -adic étale cohomology, where $\ell \neq p$ is a prime, is a reasonable choice, but it is hard to describe explicitly.

A different possibility is to choose rigid cohomology, which is a p -adic cohomology theory built in analogy to algebraic de Rham cohomology (see e.g. [5], [25]). For hypersurface complements in affine or projective space, there is a similar description as in Lemma 2.18, replacing polynomials by overconvergent power series. However, the rigid cohomology of singular varieties is a rather mysterious object. To our knowledge, it is not even known whether $H^n(X) = 0$ holds for a singular affine hypersurface $X \subseteq \mathbb{A}^n$.

The field K admits a ring of Witt vectors $W(K)$, denote its field of quotients by $Q(K)$. Let $F \in W(K)[x_0, \dots, x_n]_d$ be a homogeneous polynomial of degree d with coefficients in the ring $W(K)$. Then F defines a $W(K)$ -scheme \mathcal{X} . Its generic fiber is the hypersurface $\mathcal{X}_\eta := \{F = 0\} \subseteq \mathbb{P}_{Q(K)}^n$. The special fiber \mathcal{X}_s is a hypersurface in \mathbb{P}_K^n defined by reducing F modulo p . Both the rigid cohomology of \mathcal{X}_s and the algebraic de Rham cohomology of \mathcal{X}_η take values in $Q(K)$, and there is a natural cospecialization map relating them. This map is an isomorphism when \mathcal{X} is smooth. For singular \mathcal{X} , this is no longer true: A simple example is given by $F = x_0^2 + x_1^2 + x_2^2 + x_3^2 + px_4^2 \in \mathbb{Z}_p[x_0, \dots, x_4]$. The corresponding generic fiber \mathcal{X}_η is a smooth hypersurface in $\mathbb{P}_{\mathbb{Q}_p}^4$ and hence $h_{\mathrm{dR}}^4(\mathcal{X}_\eta) = 1$. On the other hand, $\mathcal{X}_s \subseteq \mathbb{P}_{\mathbb{F}_p}^4$ is not factorial, so $h_{\mathrm{rig}}^4(\mathcal{X}_s) > 1$ by Theorem 4.1. If instead we choose $F = x_0^2 + x_1^2 + x_2^2 + x_3^2$, the special fiber does not change, but the generic fiber has defect as well.

This motivates the following question:

Question. Let $X \subseteq \mathbb{P}_K^n$ be a hypersurface with defect in rigid cohomology. Does X admit a lift $\mathcal{X} \subseteq \mathbb{P}_{W(K)}^n$ such that the generic fiber $\mathcal{X}_\eta \subseteq \mathbb{P}_{Q(K)}^n$ has defect in algebraic de Rham cohomology?

If this question had an affirmative answer, then we could use the results of Section 2:

Corollary 2.25. *Let $X \subseteq \mathbb{P}_K^n$ be a hypersurface of degree d with global Tjurina number τ admitting a lift with defect. Then*

$$\tau \geq \frac{d - n + 1}{n^2 + n + 1}.$$

Proof. The Nakayama lemma implies that the Tjurina number cannot decrease after reduction mod p . Apply Theorem 2.21. \square

However, this question seems to be very delicate. By [35, Theorem 1.1], there are surfaces $S \subseteq \mathbb{P}^4$ that do not lift to characteristic zero. Such surfaces cannot be complete intersections, so no hypersurface X containing S can be factorial. In particular, if such an X is defined over $\overline{\mathbb{F}}_p$, then X will have defect by Theorem 4.1. On the other hand, it is well possible that every lift of X is factorial, as we cannot lift S .

3. RESOLUTION OF SINGULARITIES

3.1. Cohomological preliminaries. In this section, we relate defect of hypersurfaces to the number of singularities following the ideas presented in [31]. Let K be an algebraically closed field of characteristic $p \neq 2$. Denote by H^\bullet one of these theories:

- étale cohomology with coefficients in \mathbb{Q}_ℓ , where $\ell \neq p$ is a prime,
- algebraic de Rham cohomology with coefficients in K (if $p = 0$),
- rigid cohomology with coefficients in the field of quotients of the ring of Witt vectors of K (if $p > 0$).

All these theories feature the cohomological facts 2.3-2.9 and the Lefschetz hyperplane theorem 2.10 with the small exception that it is not known whether $H_{\text{rig}}^i(Z) = 0$ for singular affine varieties Z and $i > \dim Z$. However, this will only be used for affine hypersurfaces with weighted homogeneous singularities, where the required statements follow from [29, §3.2]. The advantage is now that we can freely use the cohomological proofs of Section 2.

Moreover, we will need two more cohomological tools.

Lemma 3.1 (Long exact sequence of a proper birational morphism). *Let X be a complete variety over K . Further let $\pi : Y \rightarrow X$ be a proper birational morphism such that its restriction $\pi|_{Y \setminus E} : Y \setminus E \rightarrow X \setminus \Sigma$ is an isomorphism for certain closed subschemes $E \subseteq Y$ and $\Sigma \subseteq X$. Then there is a long exact sequence*

$$\cdots \rightarrow H^i(X) \rightarrow H^i(Y) \oplus H^i(\Sigma) \rightarrow H^i(E) \rightarrow H^{i+1}(X) \rightarrow \cdots$$

Proof. See also [18, Theorem II.4.4] for algebraic de Rham cohomology and [23, Proposition 2.3] for étale cohomology. Since cohomology with compact support is contravariant

with respect to proper morphisms, the resolution π induces a commutative ladder

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H_c^i(X \setminus \Sigma) & \longrightarrow & H_c^i(X) & \xrightarrow{\beta} & H_c^i(\Sigma) \longrightarrow H_c^{i+1}(X \setminus \Sigma) \longrightarrow \cdots \\
& & \parallel & & \alpha \downarrow & & \gamma \downarrow & & \parallel \\
\cdots & \longrightarrow & H_c^i(Y \setminus E) & \longrightarrow & H_c^i(Y) & \xrightarrow{\delta} & H_c^i(E) \longrightarrow H_c^{i+1}(Y \setminus E) \longrightarrow \cdots
\end{array}$$

By diagram chasing, this yields a long exact sequence

$$\cdots \rightarrow H_c^i(X) \xrightarrow{(\alpha, \beta)} H_c^i(Y) \oplus H_c^i(\Sigma) \xrightarrow{\gamma - \delta} H_c^i(E) \rightarrow H_c^{i+1}(X) \rightarrow \cdots$$

Since X, Σ, Y, E are all complete, we can omit the compact support. \square

Lemma 3.2 (Mayer-Vietoris sequence). *Let X_1, \dots, X_r be projective varieties over K and let $X := X_1 \cup \cdots \cup X_r$. Suppose that the triple intersections $X_j \cap X_k \cap X_\ell$ are empty for pairwise distinct j, k, ℓ . Then there is a long exact sequence*

$$\cdots \rightarrow H^i(X) \rightarrow \bigoplus_{j=1}^r H^i(X_j) \rightarrow \bigoplus_{1 \leq j < k \leq r} H^i(X_j \cap X_k) \rightarrow H^{i+1}(X) \rightarrow \cdots$$

Proof. In the algebraic de Rham case, let $X \hookrightarrow Y$ be a closed embedding into a smooth projective variety Y . Then there is a short exact sequence of formally completed de Rham complexes

$$0 \rightarrow \Omega_Y^\bullet / X \rightarrow \bigoplus_{j=1}^r \Omega_Y^\bullet / X_j \rightarrow \bigoplus_{1 \leq j < k \leq r} \Omega_Y^\bullet / X_j \cap X_k \rightarrow 0,$$

compare [18, Proposition II.4.1]. It remains to apply hypercohomology. The proof for rigid cohomology is analogous: Embed X into the closed fiber of a smooth formal scheme \mathcal{P} and use the short exact sequence

$$0 \rightarrow \Omega_{]X[_{\mathcal{P}}}^\bullet \rightarrow \bigoplus_{j=1}^r \Omega_{]X_j[_{\mathcal{P}}}^\bullet \rightarrow \bigoplus_{1 \leq j < k \leq r} \Omega_{]X_j \cap X_k[_{\mathcal{P}}}^\bullet \rightarrow 0.$$

For étale cohomology, let ι_j resp. $\iota_{j,k}$ denote the inclusion of X_j resp. $X_j \cap X_k$ into X and take the long exact cohomology sequence of

$$0 \rightarrow \mathbb{Q}_\ell \rightarrow \bigoplus_{j=1}^r \iota_{j*} \mathbb{Q}_\ell \rightarrow \bigoplus_{1 \leq j < k \leq r} \iota_{j,k*} \mathbb{Q}_\ell \rightarrow 0. \quad \square$$

3.2. Hypersurfaces with ordinary multiple points and A_k singularities. For a positive integer $n \geq 3$, let $X \subseteq \mathbb{P}_K^n$ be an irreducible hypersurface of degree d with isolated singularities. Again, we define the *defect* of X as

$$\delta(X) := h^n(X) - h^n(\mathbb{P}^n).$$

Suppose further that the singular points belong to the following classes:

- *Ordinary multiple points.* A point x is an *ordinary multiple point of multiplicity m* if the projectivized tangent cone at x is the cone over a smooth degree m hypersurface in \mathbb{P}^{n-1} for some $m \geq 2$.

- A_k *singularities*. These are points whose completed local ring is isomorphic to

$$K[[x_1, \dots, x_n]]/(x_1^{k+1} + x_2^2 + \dots + x_n^2)$$

for some $k \geq 1$.

Note that an ordinary double point is an A_1 singularity, and this is the only common member of both families.

Let Σ_O be the set of ordinary multiple points in X of multiplicity ≥ 3 , and denote by m_x the multiplicity of a point $x \in \Sigma_O$. Similarly, define Σ_A to be the union of all A_k points in X for $k \geq 1$, and for an A_k singularity $x \in \Sigma_A$ let $r_x := \lceil k/2 \rceil$.

The advantage of restricting to these two classes of singularities is the very explicit nature of a resolution of singularities:

Proposition 3.3. *Let X be as above. Then there is an embedded resolution of singularities $\pi : (Y \subseteq P) \rightarrow (X \subseteq \mathbb{P}^n)$ such that P is a smooth n -fold obtained from \mathbb{P}^n by a finite sequence of blowups in points. More precisely:*

- (1) P is obtained by \mathbb{P}^n as a sequence of

$$s := \#\Sigma_O + \sum_{x \in \Sigma_A} r_x$$

blowups in points.

- (2) As a divisor on P , the strict transform Y of X is linearly equivalent to

$$dH - \sum_{x \in \Sigma_O} m_x \mathcal{D}_x - \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} 2i \cdot \mathcal{E}_{x,i}$$

where

- H is the pullback of a hyperplane,
- $\mathcal{D}_x \cong \mathbb{P}^{n-1}$ and $D_x := Y \cap \mathcal{D}_x$ is a smooth degree m_x hypersurface in \mathbb{P}^{n-1} ,
- $\mathcal{E}_{x,i}$ is obtained from \mathbb{P}^{n-1} by $r_x - i$ blowups in points and $E_{x,i} := Y \cap \mathcal{E}_{x,i}$ is isomorphic to the blowup at the vertex of the cone over a smooth quadric in \mathbb{P}^{n-2} for $i = 1, \dots, r_x - 1$.
- $\mathcal{E}_{x,r_x} \cong \mathbb{P}^{n-1}$ and $E_{x,r_x} := Y \cap \mathcal{E}_{x,r_x}$ is isomorphic to a smooth quadric in \mathbb{P}^{n-1} if k is odd,
- $\mathcal{E}_{x,r_x} \cong \mathbb{P}^{n-1}$ and $E_{x,r_x} := Y \cap \mathcal{E}_{x,r_x}$ is isomorphic to the cone over a smooth quadric in \mathbb{P}^{n-2} if k is even.
- $E_{x,i} \cap E_{x,j} = \emptyset$ unless $|i - j| \leq 1$ and $E_{x,i} \cap E_{x,i+1}$ is isomorphic to a smooth quadric in \mathbb{P}^{n-2} for $i = 1, \dots, r_x - 1$.
- $E_{x,i} \cap E_{x,j} \cap E_{x,k} = \emptyset$ for pairwise distinct i, j, k .

Proof. See [31] for the case of ordinary multiple points and [9], [34] for details on resolving A_k singularities. \square

3.3. Vanishing of local cohomology. Before computing Betti numbers of the resolution, we remark that if n happens to be odd, A_k singularities do not contribute to defect:

Lemma 3.4. *If n is odd, then $H_{\Sigma_A}^n(X) = 0$. In particular, if X has at most A_k singularities, then $H^n(X) = 0$.*

Proof. The “in particular” statement follows from Lemma 2.13.

The result is well-known for de Rham cohomology in characteristic zero, see [11, Examples 1.9]. In general, the space $H_{\Sigma_A}^n(X)$ decomposes into the direct sum $\bigoplus_{x \in \Sigma_A} H_{\{x\}}^n(X)$. Moreover, $H_{\{x\}}^n(X)$ depends only on (X, x) up to contact equivalence, see [30, Subsection 1.2] for étale and rigid cohomology.

Thus we are left with computing $H_{\{0\}}^n(Z)$ for the variety $Z = \{x_1^k + x_2^2 + \dots + x_n^2 = 0\} \subseteq \mathbb{A}^n$. Consider the exact sequence

$$\dots \rightarrow H^{n-1}(Z) \rightarrow H^{n-1}(Z \setminus \{0\}) \rightarrow H_{\{0\}}^n(Z) \rightarrow H^n(Z) \rightarrow \dots$$

Since $H^n(Z) = 0$, we only need to show that $H^{n-1}(Z \setminus \{0\}) = 0$. By Poincaré duality, $H^{n-1}(Z \setminus \{0\}) \cong H_c^{n-1}(Z \setminus \{0\})^\vee$. Now let $\overline{Z} \subseteq \mathbb{P}^n$ denote the projective closure of Z . Then there is a compact support Gysin sequence

$$\dots \rightarrow H^{n-2}(\overline{Z} \setminus Z) \rightarrow H_c^{n-1}(Z \setminus \{0\}) \rightarrow H_c^{n-1}(\overline{Z} \setminus \{0\}) \rightarrow H^{n-1}(\overline{Z} \setminus Z) \rightarrow H^n(Z \setminus \{0\}) \rightarrow \dots$$

As in the proof of Lemma 2.13, $H^n(Z \setminus \{0\}) = 0$. The variety $\overline{Z} \setminus Z$ is either a smooth quadric in \mathbb{P}^{n-1} ($k = 1$) or a hyperplane of multiplicity k ($k \geq 2$). In both cases, we have $h^{n-2}(\overline{Z} \setminus Z) = 0$ and $h^{n-1}(\overline{Z} \setminus Z) = 1$. Thus it suffices to show that $h_c^{n-1}(\overline{Z} \setminus \{0\}) = 1$.

To this end, observe that by Proposition 3.3, \overline{Z} has a resolution of singularities that lifts to characteristic zero. Applying proper and smooth base change (étale cohomology, [28]) or the Baldassarri-Chiarello comparison theorem (rigid cohomology, [4, Corollary 2.6]), we can reduce to the known de Rham cohomology case. \square

Remark. Ordinary multiple points of multiplicity ≥ 3 can cause defect on even-dimensional hypersurfaces: Let $X \subseteq \mathbb{P}^3$ be the projective cone over a smooth plane curve C of degree $m \geq 3$. Then $h^3(X) = h^1(C) = (m-1)(m-2) > 0$ by Corollary 2.16, so X has defect.

3.4. Defect and Betti numbers of the resolution. We will now give a cohomological criterion for defect using the embedded resolution of singularities π from Proposition 3.3. First, we need the Betti numbers of P , which is obtained by s successive blowups.

Lemma 3.5. *We have*

$$h^i(P) = \begin{cases} s+1 & \text{if } i \in \{2, \dots, 2n-2\}, \\ 1 & \text{if } i \in \{0, 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $P_0 := \mathbb{P}^n$ and for $j = 1, \dots, s$ denote by P_j the blowup of P_{j-1} in a point. By Lemma 3.1, there is an exact sequence

$$\dots \rightarrow H^i(P_j) \rightarrow H^i(P_{j+1}) \oplus H^i(\{\text{pt}\}) \rightarrow H^i(\mathbb{P}^{n-1}) \rightarrow H^{i+1}(P_j) \rightarrow \dots$$

Using the Betti numbers of projective space, the claim follows by induction. \square

The next step is to compute some Betti numbers of the exceptional divisor E associated to the resolution $\pi|_Y : Y \rightarrow X$, i.e.,

$$E := Y \cap \left(\sum_{x \in \Sigma_O} D_x + \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} E_{x,i} \right).$$

Lemma 3.6. *Suppose that n is even. Then $h^{n-1}(E) = 0$ and $h^n(E) = s$.*

Proof. E is the disjoint union of the divisors D_x , $x \in \Sigma_O$, and $E_x = \sum_{i=1}^{r_x} E_{x,i}$, $x \in \Sigma_A$. Hence we can treat each singularity type separately.

- D_x for $x \in \Sigma_O$. By the description given in Proposition 3.3, D_x is isomorphic to a smooth degree m_x hypersurface in \mathbb{P}^{n-1} . Hence by Corollary 2.11, $h^i(D_x) = h^i(\mathbb{P}^{n-1})$ for $i \notin \{n-2, 2n-2\}$. In particular $h^{n-1}(D_x) = 0$ and $h^n(D_x) = 1$.
- E_x for $x \in \Sigma_A$. Let Q be a smooth quadric in \mathbb{P}^{n-2} , let C be the cone over Q in \mathbb{P}^{n-1} and denote by B the blowup of C in its vertex. Further let S be a smooth quadric in \mathbb{P}^{n-1} . Using Corollary 2.11 and Lemma 2.12, one computes that

$$h^i(Q) = h^i(C) = h^i(B) = h^i(S) = 0 \quad \text{for all odd } i \geq n-1.$$

Since there are no triple intersections between the components of E_x , Lemma 3.2 yields a long exact Mayer-Vietoris sequence

$$\cdots \rightarrow H^q(E_x) \rightarrow \bigoplus_i H^q(E_{x,i}) \xrightarrow{d_q} \bigoplus_{i < j} H^q(E_{x,i} \cap E_{x,j}) \rightarrow H^{q+1}(E_x) \rightarrow \cdots$$

We claim that the maps d_{n-2} and d_n are surjective. Assuming this, we immediately have $h^{n-1}(E_x) = 0$ by the description given in Proposition 3.3.

In the case $n = 4$, the $E_{x,i}$ are irreducible surfaces, so $h^4(E_x) = \sum_{i=1}^{r_x} h^4(E_{x,i}) = r_x$. For $n \geq 6$, one computes $h^n(Q) = h^n(S) = 1$ by Corollary 2.11, $h^n(C) = 1$ by Lemma 2.12 and thus $h^n(B) = 2$. Therefore

$$h^n(E_x) = (r_x - 1) \cdot h^n(E_{x,i}) + h^n(E_{x,r_x}) - (r_x - 1) \cdot h^n(E_{x,i} \cap E_{x,j}) = r_x.$$

It remains to prove the surjectivity of

$$\bigoplus_i H^q(E_{x,i}) \rightarrow \bigoplus_{i < j} H^q(E_{x,i} \cap E_{x,j})$$

for $q = n-2, n$. Since $E_{x,i} \cap E_{x,j}$ is empty unless $|i - j| = 1$, this would follow from the surjectivity of all the maps

$$H^q(E_{x,i}) \rightarrow H^q(E_{x,i} \cap E_{x,i+1}), \quad i = 1, \dots, r_x - 1.$$

But the intersection $E_{x,i} \cap E_{x,i+1} \cong Q$ is a smooth quadric inside the exceptional divisor $F \cong \mathbb{P}^{n-2}$ of the blowup of C at its vertex. Thus the restriction morphism $H^q(F) \rightarrow H^q(Q)$ is surjective for $q = n-2, n$. Moreover, Lemma 3.1 yields that there is an exact sequence

$$\cdots \rightarrow H^q(C) \rightarrow H^q(B) \rightarrow H^q(F) \rightarrow H^{q+1}(C) \rightarrow \cdots$$

Using $h^{q+1}(C) = 0$, we obtain that the map $H^q(B) \rightarrow H^q(F)$ is surjective and so is the composition

$$H^q(E_{x,i}) \xrightarrow{\sim} H^q(B) \rightarrow H^q(F) \rightarrow H^q(Q) \xrightarrow{\sim} H^q(E_{x,i} \cap E_{x,i+1}).$$

Summing up,

$$h^{n-1}(E) = 0 \quad \text{and} \quad h^n(E) = \sum_{x \in \Sigma_O} 1 + \sum_{x \in \Sigma_A} r_x = s. \quad \square$$

Lemma 3.7. *Suppose that n is odd. Then $h^n(E) = 0$ and $h^n(X) \leq h^n(Y)$.*

Proof. The proof that $h^n(E) = 0$ is analogous to the proof of $h^{n-1}(E) = 0$ given in Lemma 3.6.

It remains to show the inequality $h^n(X) \leq h^n(Y)$. We first blow up the ordinary multiple points successively. This gives a partial resolution $\psi : Y_O \rightarrow X$ with Y_O having at most A_k singularities. The morphism ψ comes from an embedded resolution $P_O \rightarrow \mathbb{P}^n$, and thus we have the following commutative diagram by Lemma 3.1:

$$\begin{array}{ccccccc} H^{n-1}(P_O) & \longrightarrow & \bigoplus_{x \in \Sigma_O} H^{n-1}(\mathcal{D}_x) & \longrightarrow & H^n(\mathbb{P}^n) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ H^{n-1}(Y_O) & \longrightarrow & \bigoplus_{x \in \Sigma_O} H^{n-1}(D_x) & \longrightarrow & H^n(X) & \longrightarrow & H^n(Y_O) \longrightarrow H^n(E), \end{array}$$

Since $\mathcal{D}_x \cong \mathbb{P}^{n-1}$, and D_x is a smooth hypersurface therein, the natural restriction map $H^{n-1}(\mathcal{D}_x) \rightarrow H^{n-1}(D_x)$ is an isomorphism. Together with $H^n(\mathbb{P}^n) = 0$ this implies that $H^{n-1}(Y_O) \rightarrow \bigoplus_{x \in \Sigma_O} H^{n-1}(D_x)$ is surjective and thus $H^n(X) \cong H^n(Y_O)$.

Let $D_O := \sum_{x \in \Sigma_O} D_x$. Then

$$H_{\psi^{-1}(\Sigma_A)}^n(Y_O) \cong H_{\psi^{-1}(\Sigma_A)}^n(Y_O \setminus D_O) \cong H_{\Sigma_A}^n(X \setminus \Sigma_O) = H_{\Sigma_A}^n(X) = 0$$

by Lemma 3.4. Resolving Y_O , we obtain our smooth hypersurface Y in P with the exceptional divisor $E_A \cong \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} E_{x,i}$. This resolution gives a commutative diagram

$$\begin{array}{ccccc} H_{\psi^{-1}(\Sigma_A)}^n(Y_O) & \longrightarrow & H^n(Y_O) & \longrightarrow & H^n(Y_O \setminus \psi^{-1}(\Sigma_A)) \\ & & \downarrow & & \parallel \\ & & H^n(Y) & \longrightarrow & H^n(Y \setminus E_A) \end{array}$$

It follows that $H^n(Y_O) \rightarrow H^n(Y_O \setminus \psi^{-1}(\Sigma_A)) \rightarrow H^n(Y \setminus E_A)$ is injective. This implies that the map $H^n(Y_O) \rightarrow H^n(Y)$ is injective as well.

Consequently, $h^n(X) = h^n(Y_O) \leq h^n(Y)$. □

With the two lemmas above, we obtain a simple formula for the defect of X :

Proposition 3.8. *The defect of X may be computed as follows:*

$$\delta(X) = \begin{cases} h^n(Y) - s - 1 & \text{if } n \text{ is even,} \\ h^n(Y) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Applying Lemma 3.1 to $\pi|_Y : Y \rightarrow X$, there is a long exact sequence

$$\dots \rightarrow H^{n-1}(Y) \rightarrow H^{n-1}(E) \rightarrow H^n(X) \rightarrow H^n(Y) \rightarrow H^n(E) \rightarrow H^{n+1}(X) \rightarrow \dots$$

Suppose first that n is even. Using Lemma 3.6, $h^{n-1}(E) = 0$ and $h^n(E) = s$. By Lemma 2.12 we have $h^{n+1}(X) = 0$. It follows that $h^n(Y) = h^n(X) + s$. If n is odd, then inserting $h^n(E) = 0$ into the above long exact sequence implies $h^n(X) \geq h^n(Y)$. On the other hand, $h^n(X) \leq h^n(Y)$ by Lemma 3.7, so that $h^n(X) = h^n(Y)$. □

3.5. Defect and ampleness of the strict transform. We keep the notation of the previous subsection. If the strict transform Y of X happens to be an ample divisor in P , then - embedding P into a suitable projective space - the Lefschetz hyperplane theorem 2.10 shows that the restriction map

$$H^{n-2}(P) \rightarrow H^{n-2}(Y)$$

is an isomorphism. Applying Poincaré duality on Y , $h^{n-2}(P) = h^n(Y)$. Hence we have the following corollary of Proposition 3.8 and Lemma 3.5:

Corollary 3.9. *Suppose that Y is ample in P . Then $\delta(X) = 0$.*

Finally, we can relate ampleness of Y to the number of singularities of X .

Lemma 3.10. *Suppose that*

$$\sum_{x \in \Sigma_O} m_x + \sum_{x \in \Sigma_A} 2r_x < d.$$

Then Y is ample in P .

Proof. This is a variant of [31, Theorem 4.1]. By Proposition 3.3, inside $\text{Pic}(P)$,

$$\begin{aligned} Y &= dH - \sum_{x \in \Sigma_O} m_x D_x - \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} 2i \cdot \mathcal{E}_{x,i} \\ &= dH - \sum_{x \in \Sigma_O} m_x D_x - 2 \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} \underbrace{\sum_{j=i}^{r_x} \mathcal{E}_{x,j}}_{=: \widetilde{\mathcal{E}}_{x,i}} \\ &= \left(d - \sum_{x \in \Sigma_O} m_x - \sum_{x \in \Sigma_A} 2r_x \right) H + \sum_{x \in \Sigma_O} m_x (H - D_x) + 2 \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} (H - \widetilde{\mathcal{E}}_{x,i}). \end{aligned}$$

Since H is the pullback of a hyperplane, the linear system $|H|$ has no base points. Using the hypothesis,

$$\left| \left(d - \sum_{x \in \Sigma_O} m_x - \sum_{x \in \Sigma_A} 2r_x \right) H \right|$$

is base-point free as well. If $x \in X$ is a singular point, then x is scheme-theoretically cut out by hyperplanes. In particular, its ideal sheaf twisted by $\mathcal{O}(1)$ is globally generated, and so are the pullbacks $\mathcal{O}_P(H - D_x)$ and $\mathcal{O}_P(H - \widetilde{\mathcal{E}}_{x,1})$, respectively. Similarly, $\mathcal{O}_P(H - \widetilde{\mathcal{E}}_{x,i})$ is globally generated for any i . In total, $\mathcal{O}_P(Y)$ is a globally generated invertible sheaf on P .

It follows that if $C \subseteq P$ is an irreducible curve, then $Y.C \geq 0$. In order to show that Y is ample, it suffices to show that such an intersection $Y.C$ is always positive. If $\pi_* C$ is a curve on \mathbb{P}^n , then by the projection formula

$$H.C = (\pi_* C). \mathcal{O}(1) > 0,$$

thus $Y.C > 0$.

If C is contracted by π , then $H.C = 0$ again by the projection formula. By base-point freeness of $|H - D_x|$ and $|H - \widetilde{\mathcal{E}}_{x,i}|$, all the intersection numbers $D_x.C$ and $\widetilde{\mathcal{E}}_{x,i}.C$ are hence

nonpositive. The Picard group of P is spanned by H , the \mathcal{D}_x and the $\widetilde{\mathcal{E}}_{x,i}$. Since P is projective, there must be integers $h, d_x, e_{x,i}$ such that the divisor

$$A := hH + \sum_{x \in \Sigma_O} d_x \mathcal{D}_x + \sum_{x \in \Sigma_A} e_{x,i} \widetilde{\mathcal{E}}_{x,i}$$

is ample and thus $A.C > 0$. In particular, at least one of the intersection products $D_x.C$ or $\widetilde{\mathcal{E}}_{x,i}.C$ is nonzero and hence strictly negative. This implies $Y.C > 0$.

Consequently, Y is ample in P . \square

Remark. This proof does not carry over to singular points of type D_k or E_k . For $n = 4$, the standard embedded resolution of these singularities has the property that the s exceptional divisors of the resolution $P \rightarrow \mathbb{P}^n$ break into several components when intersecting with the strict transform Y of X . In particular, $h^4(E) \geq s + 1 = h^2(P)$. But then by Lemma 3.1

$$h^4(Y) \geq h^4(X) + h^4(E) \geq h^4(X) + h^2(P) \geq 1 + h^2(P),$$

thus $h^4(Y) = h^2(Y) \neq h^2(P)$. Consequently, Y cannot be ample in P in virtue of the Lefschetz hyperplane theorem.

However, in case that the ground field is of characteristic zero, the Hodge numbers of resolutions of hypersurfaces with at most ADE singularities were investigated by Rams [33, §4].

3.6. Proof of Theorem 1.2.

Proof. Suppose that X has defect. Let $\pi : (Y \subseteq P) \rightarrow (X \subseteq \mathbb{P}^n)$ be the embedded resolution from Proposition 3.3. By Corollary 3.9, Y cannot be ample in P . Now Lemma 3.10 implies that

$$\sum_{x \in \Sigma_O} m_x + \sum_{x \in \Sigma_A} 2r_x \geq d. \quad \square$$

4. FACTORIAL THREEFOLD HYPERSURFACES OVER $\overline{\mathbb{F}_p}$

Let K be a field and $X \subseteq \mathbb{P}_K^4$ be a hypersurface defined by a homogeneous polynomial $f \in K[x_0, \dots, x_4]$.

X is *factorial* if the homogeneous coordinate ring $K[x_0, \dots, x_4]/(f)$ is a unique factorization domain. By [19, Exercise II.6.3], X is factorial if and only if the natural map $\text{Pic}(X) \rightarrow \text{Cl}(X)$ is an isomorphism, i.e., if and only if every Weil divisor on X is linearly equivalent to a Cartier divisor.

Furthermore, X is called \mathbb{Q} -*factorial* if the map $\text{Pic}(X) \rightarrow \text{Cl}(X)$ becomes an isomorphism after tensoring with \mathbb{Q} , i.e. if every Weil divisor on X is linearly equivalent to a \mathbb{Q} -Cartier divisor.

Theorem 4.1. *Suppose $K \subseteq \overline{\mathbb{F}_p}$. Let $X \subseteq \mathbb{P}_K^4$ be a hypersurface with at most isolated singularities. If $h_{\text{ét}}^4(X) = 1$ or $h_{\text{rig}}^4(X) = 1$, then X is factorial.*

Remark. The corresponding statement in characteristic zero is shown in [31, Proposition 3.2].

From now on, let $X \subseteq \mathbb{P}_K^4$ be a hypersurface defined over $K = \overline{\mathbb{F}_p}$ with zero-dimensional singular locus Σ . Since X is a threefold, [8] provides a resolution of singularities $\pi : Y \rightarrow X$. Denote by E the exceptional divisor and by s the number of its irreducible components.

Lemma 4.2. *With the above notations,*

$$\mathrm{rk} \, \mathrm{Cl}(X) = \mathrm{rk} \, \mathrm{Pic}(Y) - s.$$

Proof. Since Σ has codimension at least two in X ,

$$\mathrm{Cl}(X) \cong \mathrm{Cl}(X \setminus \Sigma) \cong \mathrm{Cl}(Y \setminus E).$$

Let E_1, \dots, E_s denote the irreducible components of the exceptional divisor E . Then there is a standard exact sequence

$$\bigoplus_{i=1}^s \mathbb{Z} \cdot E_i \rightarrow \mathrm{Cl}(Y) \rightarrow \mathrm{Cl}(Y \setminus E) \rightarrow 0.$$

This sequence is also exact on the left: Suppose $\sum_{i=1}^s a_i [E_i] = 0 \in \mathrm{Cl}(Y)$ for $a_1, \dots, a_s \in \mathbb{Z}$. If $H \subseteq Y$ is general hyperplane, then $D := \sum_{i=1}^s a_i (E_i \cap H)$ is linearly equivalent to 0 as a divisor on the surface $Y \cap H$. However, as in [15, Example 2.4.4], D has negative self-intersection, contradicting that $[D] = 0 \in \mathrm{Cl}(Y \cap H)$. \square

Lemma 4.3. *For both étale and rigid cohomology,*

$$h^4(Y) - s \leq h^4(X).$$

Proof. Since $H^4(\Sigma) = 0$ as $\dim \Sigma = 0$, this follows from the long exact sequence

$$\dots \rightarrow H^4(X) \rightarrow H^4(Y) \oplus H^4(\Sigma) \rightarrow H^4(E) \rightarrow \dots$$

Lemma 3.1. \square

In order to compare Picard rank and Betti numbers, we need the following result on the étale cycle class map:

Lemma 4.4. *Let Z be a smooth projective variety over K . Then the étale cycle class map*

$$\mathrm{Pic}(Z) \otimes \mathbb{Q}_\ell \rightarrow H^2(Z, \mathbb{Q}_\ell(1))$$

is injective.

Proof. Let ℓ be a prime not equal to $\mathrm{char}(K)$. The étale cycle class map tensored with \mathbb{Q}_ℓ factors as

$$\mathrm{Pic}(Z) \otimes \mathbb{Q}_\ell \xrightarrow{\alpha} \mathrm{NS}(Z) \otimes \mathbb{Q}_\ell \xrightarrow{\beta} H^2(Z, \mathbb{Q}_\ell(1)),$$

where $\mathrm{NS}(Z)$ denotes the Néron-Severi group of Z . As in [28, pp. 216–217], one obtains that β is injective. The kernel of α is precisely $\mathrm{Pic}^0(Z) \otimes \mathbb{Q}_\ell$. But since $K = \overline{\mathbb{F}_p}$, the group $\mathrm{Pic}^0(Z)$ is torsion [22, Lemma 2.16]. Hence α is injective as well. \square

Corollary 4.5. *For both étale and rigid cohomology, we have $\mathrm{rk} \, \mathrm{Cl}(X) \leq h^4(X)$. In particular, if $h^4(X) = 1$, then X is \mathbb{Q} -factorial.*

Proof. By Lemma 4.4, $\mathrm{rk} \mathrm{Pic}(Y) \leq h_{\mathrm{\acute{e}t}}^2(Y, \mathbb{Q}_\ell(1)) = h_{\mathrm{\acute{e}t}}^2(Y, \mathbb{Q}_\ell)$. As étale and rigid cohomology are both Weil cohomologies and Y is defined over some finite field, applying the comparison theorem of Katz-Messing [21, Corollary 1] yields $h_{\mathrm{\acute{e}t}}^2(Y) = h_{\mathrm{rig}}^2(Y)$. Now Poincaré duality on Y gives $h^2(Y) = h^4(Y)$. Thus, with the help of Lemma 4.2 and Lemma 4.3,

$$\mathrm{rk} \mathrm{Cl}(X) = \mathrm{rk} \mathrm{Pic}(Y) - s \leq h^4(Y) - s \leq h^4(X). \quad \square$$

Finally, we need to proceed from \mathbb{Q} -factoriality to factoriality.

Lemma 4.6. *If X is \mathbb{Q} -factorial, then X is factorial.*

Proof. We follow the proof of [31, Proposition 2.15]. Since X is normal and Cohen-Macaulay, the proof of [20, Proposition 2.15] generalizes and gives an exact sequence

$$0 \rightarrow \mathrm{Pic}(X) \rightarrow \mathrm{Cl}(X) \rightarrow \bigoplus_{x \in \Sigma} \mathrm{Cl}(\mathcal{O}_{X,x}).$$

In particular, there is an injection

$$\mathrm{Cl}(X)/\mathrm{Pic}(X) \hookrightarrow \bigoplus_{x \in \Sigma} \mathrm{Cl}(\mathcal{O}_{X,x}).$$

By hypothesis, $\mathrm{Cl}(X)/\mathrm{Pic}(X)$ is a torsion group. Fix $x \in \Sigma$. By [10, Corollary 2.10], the Picard group of the punctured spectrum U_x of $\mathcal{O}_{X,x}$ is torsion-free. Since X has only isolated singularities, $\mathrm{Pic}(U_x) \cong \mathrm{Cl}(\mathcal{O}_{X,x})$, see [14, Proposition 18.10]. Consequently, $\mathrm{Cl}(X)/\mathrm{Pic}(X)$ is a torsion subgroup of a torsion-free group and hence trivial. Thus X is factorial. \square

Proof of Theorem 4.1. Since étale and rigid cohomology behave well with respect to base change, $X \times_{\mathrm{Spec} K} \mathrm{Spec} \overline{\mathbb{F}_p}$ is factorial by Corollary 4.5 and Lemma 4.6. In other words, if S denotes the homogeneous coordinate ring of X , then $S \otimes_K \overline{\mathbb{F}_p}$ is factorial. But this implies that S and hence X are factorial. \square

5. DENSITY OF HYPERSURFACES WITHOUT DEFECT

Let $K = \mathbb{F}_q$ be a finite field of characteristic $\neq 2$. By a result of Poonen, the asymptotic density of smooth hypersurfaces in \mathbb{P}^n defined over K is computed as follows:

Theorem 5.1 (Poonen’s Bertini theorem, [32, Theorem 1.1]).

$$\lim_{d \rightarrow \infty} \frac{\#\{f \in K[x_0, \dots, x_n]_d \mid \{f = 0\} \subseteq \mathbb{P}_K^n \text{ is smooth}\}}{\#K[x_0, \dots, x_n]_d} = \frac{1}{\zeta_{\mathbb{P}_K^n}(n+1)}.$$

Here, $\zeta_{\mathbb{P}_K^n}$ denotes the Hasse-Weil zeta function of \mathbb{P}_K^n , which is simply given by

$$\zeta_{\mathbb{P}_K^n}(s) = \prod_{i=1}^n (1 - q^{i-s}) \quad \text{for } s \in \mathbb{C}, \quad \mathrm{Re}(s) > n.$$

One trivial remark is that the limit in Theorem 5.1 is smaller than 1, so that a “random” hypersurface is smooth with a probability strictly less than 100%. However, it is true that hypersurfaces with few singularities compared to the degree form a set of density 1:

Theorem 5.2 ([26, Corollary 5.9]). *Fix a constant $c > 0$. Then*

$$\lim_{d \rightarrow \infty} \frac{\#\{f \in K[x_0, \dots, x_n]_d \mid \tau(f) \leq c \cdot d\}}{\#K[x_0, \dots, x_n]_d} = 1,$$

where $\tau(f)$ denotes the global Tjurina number of the hypersurface $\{f = 0\} \subseteq \mathbb{P}_K^n$.

If Theorem 1.1 held over finite fields, then this would imply that hypersurfaces without defect form a set of density 1. However, so far, we can only use the restricted singularity types from Theorem 1.2.

Lemma 5.3. *Let $x \in \mathbb{A}_K^n$ be a closed point with residue field $\kappa(x)$. Fix a positive integer d and choose a polynomial $f \in K[x_1, \dots, x_n]_{\leq d}$ uniformly at random. Then the probability that $\{f = 0\}$ has at most an A_k singularity for some $k \geq 1$ in x is at least*

$$1 - \#\kappa(x)^{-n-3}.$$

Proof. Let \mathcal{O}_x be the local ring of \mathbb{A}_K^n at x and denote by \mathfrak{m}_x its maximal ideal. Let

$$[f] = f_0 + f_1 + f_2 \in \mathcal{O}_x/\mathfrak{m}_x^3 \quad \text{with } \deg f_i = i, \quad i = 0, 1, 2,$$

be the 2-jet of f at x . Define X to be the hypersurface $\{f = 0\} \subseteq \mathbb{A}_K^n$. Then:

- (1) X does not pass through $x \Leftrightarrow f_0 \neq 0$,
- (2) X is smooth at $x \Leftrightarrow f_0 = 0$ and $f_1 \neq 0$,
- (3) X has an ordinary double point at $x \Leftrightarrow f_0 = 0$, $f_1 = 0$ and f_2 is a quadratic form of rank n ,
- (4) X has an A_k singularity for some $k \geq 2$ at $x \Leftrightarrow f_0 = 0$, $f_1 = 0$ and f_2 is a quadratic form of rank $n - 1$.

The vector space $\mathcal{O}_x/\mathfrak{m}_x^3$ has dimension $1 + n + \frac{n(n+1)}{2}$ over $\kappa(x)$. Let $r := \#\kappa(x)$. The probability that X has at most an A_k singularity at x hence equals

$$\frac{(r-1)r^{n+n(n+1)/2} + (r^n - 1)r^{n(n+1)/2} + p_{n,r}}{r^{1+n+n(n+1)/2}} = 1 - \frac{r^{n(n+1)/2} - p_{n,r}}{r^{1+n+n(n+1)/2}}.$$

where $p_{n,r}$ is the number of quadratic forms in n variables of rank $\geq n - 1$ over $\kappa(x)$. The bounds from the subsequent Lemma 5.4 give

$$1 - r^{-n-4} \geq 1 - \frac{r^{n(n+1)/2} - p_{n,r}}{r^{1+n+n(n+1)/2}} \geq 1 - r^{-n-3}. \quad \square$$

Lemma 5.4. *The number $p_{n,q}$ of quadratic forms in n variables of rank $\geq n - 1$ over a field with q elements equals*

$$p_{n,q} = \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} \frac{q^{2i}}{q^{2i} - 1} \prod_{i=0}^{n-2} (q^{n-i} - 1) - \prod_{i=1}^{\lfloor n/2 \rfloor} \frac{q^{2i}}{q^{2i} - 1} \prod_{i=0}^{n-1} (q^{n-i} - 1).$$

Moreover,

$$q^{\frac{n(n+1)}{2}}(1 - q^{-2}) \leq p_{n,q} \leq q^{\frac{n(n+1)}{2}}(1 - q^{-3}).$$

Proof. The formula for $p_{n,q}$ can be found in [27, Theorem 2]. Suppose first that n is even. Then

$$\begin{aligned}
p_{n,q} &= \left(1 + \frac{q^n}{q^n - 1} \cdot (q - 1)\right) \prod_{i=1}^{n/2-1} \frac{q^{2i}}{q^{2i} - 1} \prod_{i=0}^{n-2} (q^{n-i} - 1) \\
&= \frac{q^{n+1} - 1}{q^n - 1} \cdot \prod_{i=1}^{n/2-1} \frac{q^{2i}}{q^{2i} - 1} \prod_{i=0}^{n-2} (q^{n-i} - 1) \\
&= \prod_{i=1}^{n/2-1} q^{2i} \cdot \prod_{i=0}^{n/2-1} (q^{n+1-2i} - 1) \\
&= \prod_{i=0}^{n/2-1} (q^{n+1} - q^{2i}) \\
&= q^{\frac{n(n+1)}{2}} \prod_{i=0}^{n/2-1} (1 - q^{2i-n-1}) \\
&= q^{\frac{n(n+1)}{2}} (q^{-n-1}; q^2)_{n/2},
\end{aligned}$$

where we used the notation for the q -Pochhammer symbol. It is clear that $(q^{-n-1}; q^2)_{n/2}$ is a decreasing sequence bounded above from $1 - q^{-3}$. Induction on $q \geq 2$ shows the inequality

$$\prod_{i=3}^n (1 - q^{-i}) \geq 1 - q^{-2} + q^{-n},$$

whence

$$(q^{-n-1}; q^2)_{n/2} \geq \prod_{i=3}^{\infty} (1 - q^{-i}) \geq 1 - q^{-2}.$$

For odd n , we can reduce to the even case by observing that $p_{n,q} = q^n \cdot p_{n-1,q}$. □

The following proves Theorem 1.3:

Corollary 5.5. *Let K be a finite field of odd characteristic. Then*

$$\lim_{d \rightarrow \infty} \frac{\#\{f \in K[x_0, \dots, x_n]_d \mid \{f = 0\} \subseteq \mathbb{P}_K^n \text{ has no defect}\}}{\#K[x_0, \dots, x_n]_d} \geq \frac{1}{\zeta_{\mathbb{P}_K^n}(n+3)}.$$

Proof. For a property P of hypersurfaces defined by polynomials $f \in K[x_0, \dots, x_n]_d$, write

$$\mu(P) := \lim_{d \rightarrow \infty} \frac{\#\{f \in K[x_0, \dots, x_n]_d \mid \{f = 0\} \subseteq \mathbb{P}_K^n \text{ satisfies } P\}}{\#K[x_0, \dots, x_n]_d}.$$

By Theorems 1.2 and 5.2, there is a constant $c > 0$ such that

$$\mu(\text{defect and at most } A_k \text{ singularities}) \leq \mu(\tau(f) > c \cdot d) = 0.$$

Moreover, combining Lemma 5.3 with [32, Theorem 1.3],

$$\begin{aligned} \mu(\text{defect and worse than } A_k \text{ singularities}) &\leq \mu(\text{worse than } A_k \text{ singularities}) \\ &\leq 1 - \frac{1}{\zeta_{\mathbb{P}_K^n}(n+3)}. \end{aligned}$$

Putting this together,

$$\begin{aligned} \mu(\text{no defect}) &= 1 - \mu(\text{defect}) \\ &= 1 - \mu(\text{defect and at most } A_k \text{ sing.}) - \mu(\text{defect and worse than } A_k \text{ sing.}) \\ &\geq \frac{1}{\zeta_{\mathbb{P}_K^n}(n+3)}, \end{aligned}$$

which completes the proof. \square

Remark. In view of Theorem 1.2, we could have added the contribution of ordinary multiple points. The probability for a hypersurface to have a singularity at a point x and this being an ordinary multiple point of multiplicity ≥ 3 , equals

$$\sum_{d \geq 3} \#\{f \in \kappa(x)[x_1, \dots, x_n]_d \mid \{f = 0\} \text{ is smooth}\} \cdot \#\kappa(x)^{-\binom{n+d}{d}}.$$

This turns out to be small compared to the local density of at most A_k singularities and we do not expect this to bring a substantial improvement to the bound given in Lemma 5.3.

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